NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Extension 1 Mathematics Assessment Task 1
Term 4, 2013

Name:________________________________ **Mathematics Class: 11Mx**_______

Student Number: __________________________

Time Allowed: **55 minutes + 2 minutes reading time**

Total Marks: **42**

Instructions:

- Attempt all questions.
- Start each question in a new booklet. Put your name on every booklet.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Work down the page.
- Do not work in columns.
- Each question will be collected separately. Submit a blank booklet if you do not attempt a question.

Question	1	2	3-5	6	7	8 a	8 b,c	Total
PE3	/1			/6	/3		/7	/17
PE4		/1		/7				/8
HE7			/3		/9	/5		/17
	/1	/1	/3	/13	/12	/5	/7	/42

Section I

5 marks

Attempt Questions 1 - 5

Use the multiple choice answer sheet for Questions $1-5$

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1 One of the factors of $P(x)=ax^3-7x^2+kx+4$ is $(x-4)$ and the remainder when $P(x)$ is divided by $(x-1)$ is -6.

Which of the following is correct?

(A) $16a+k=-27$ and $a+k=-3$

(B) $16a+k=27$ and $a+k=3$

(C) $16a+k=-27$ and $a+k=3$

(D) $16a+k=27$ and $a+k=-3$

2 Two points $P(2t,t^2)$ and $Q(2s,s^2)$ lie on the parabola $x^2=4y$. It is known that $ts=-4$. What are the coordinates of the midpoint of PQ?

A) $\left(\dfrac{-8}{t},\ \dfrac{16}{t^2}\right)$

B) $\left(\dfrac{t^2-4}{t},\ \dfrac{t^4+16}{2t^2}\right)$

C) $\left(-t,\ \dfrac{17t^2}{2}\right)$

D) $\left(\dfrac{2t^2-4}{t},\ \dfrac{t^4+16}{t^2}\right)$

3 The derivative of a function is given as $f'(x)=\dfrac{x^2}{x-2}$. Which of the following could be true of the original function $y=f(x)$ at the point where $x=0$?

A) There is a horizontal point of inflection

B) There is a minimum turning point

C) There is a maximum turning point

D) The gradient is not defined

4 The chord of contact from an external point $A(x_0, y_0)$ to the general parabola $x^2 = 4ay$ has equation $xx_0 = 2a(y + y_0)$. From what external point are the tangents to the parabola $x^2 = 6y$ to be drawn so that $2x - 3y - 3 = 0$ is the chord of contact?

A) $(1, 1)$

B) $(2, 1)$

C) $(1, -3)$

D) $(2, -3)$

5 What is the derivative of $\dfrac{x}{\sqrt{1-2x}}$?

A) $\dfrac{1-x}{(1-2x)\sqrt{1-2x}}$

B) $\dfrac{1+x}{\sqrt{1-2x}}$

C) $\dfrac{1-3x}{(1-2x)\sqrt{1-2x}}$

D) $\dfrac{1+x-2x^2}{\sqrt{1-2x}}$

Section II

38 marks
Attempt Questions $6-8$

Answer each question in a SEPARATE writing booklet. Extra writing paper is available.

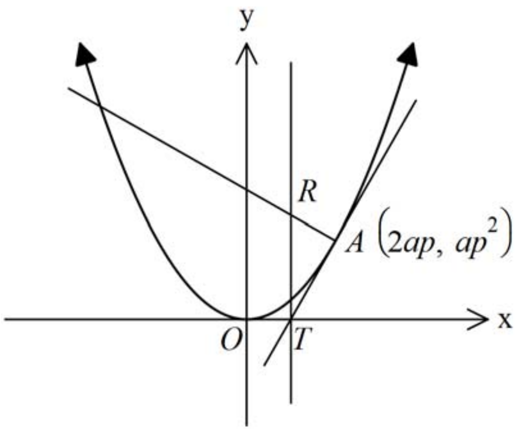
In Questions $6-8$, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (13 marks) Use a SEPARATE writing booklet.

a) By writing down factors, or otherwise, construct a monic polynomial of the form x^3+bx^2+cx+d which has zeros $1,\ 2+\sqrt{2},\ 2-\sqrt{2}$. **3**

b) Write x^3-x^2-4x in the form $A(x-1)^3+B(x-1)^2+C$ **3**

c) The parabola has parametric equations $x=2at$ and $y=at^2$. A tangent and normal have been drawn at a variable point $A\left(2ap,\ ap^2\right)$ which lies on the parabola.



(i) Show that the equation of the tangent at the point A is $y=px-ap^2$ **2**

(ii) The tangent cuts the x axis at T. Find the coordinates of T. **1**

(iii) The equation of the normal at A is $x+py=2ap+ap^3$. Do NOT show this.
A line through T parallel to the axis of the parabola cuts the normal at R. Show that the coordinates of R are $\left(ap,\ ap^2+a\right)$. **1**

(iv) Show that the locus of R is a parabola and state the equation of its directrix. **3**

Question 7 (12 marks) Use a SEPARATE booklet.

a) Solve $x^3 - 2x^2 - 7x + 2 = 0$. **3**

b) Let $P(x) = -2x^3 + kx^2 - mx + 5$. Show that if $P(x)$ is to have any stationary points, then $k^2 - 6m \geq 0$ **2**

c) Consider the curve given by the equation $y = \dfrac{(x-1)^2}{x+1}$.

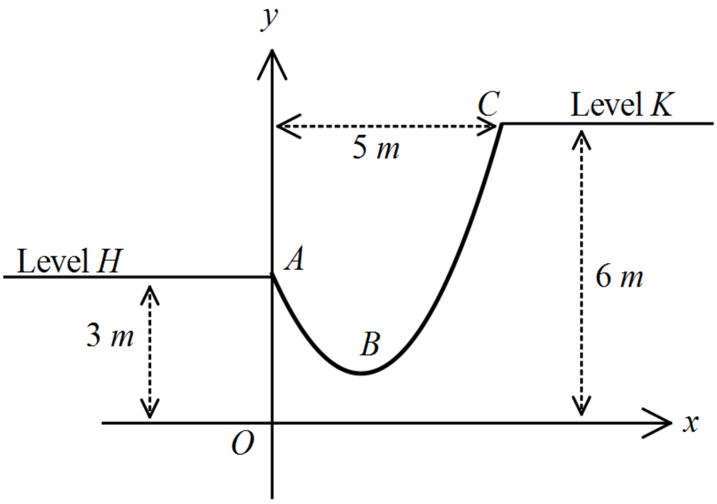
(i) Show that there are stationary points at $x = -3$ and $x = 1$, and determine their nature. **3**

(ii) Show that $\dfrac{(x-1)^2}{x+1} = x - 3 + \dfrac{4}{x+1}$ **1**

(iii) Sketch the curve showing all important features. **3**

Question 8 (12 marks) Use a SEPARATE booklet

a) The city council has decided to build a skateboard ramp for its teenagers. The structure will consist of two levels, H and K and a ramp, as shown in the diagram. The engineers believe that if the ramp has a gradient of greater than 3 at any point, it will be too dangerous to use.
A cross-section of the proposed ramp is shown below.



The ramp ABC is given by the equation $y=\frac{8x^2}{15}-\frac{31x}{15}+3$ for $0\le x\le 5$.

(i) Use the information given in the cross-section to write down the coordinates of the point C. The point A is $(0,3)$. **1**

(ii) Determine the maximum gradient of the ramp over this domain. **2**

(iii) What is the greatest height that level K may be constructed so that the ramp is deemed safe? Give your answer correct to 1 decimal place. **2**

b) The parametric equations of a curve are $x=t^2+1$ and $y=\frac{1}{t}$.

Without eliminating t,

(i) show that $\frac{dy}{dx}=\frac{1}{2}$ at $t=-1$ **2**

(ii) find the equation of the tangent at the point where $t=-1$ **1**

c) (i) If $P(x)=ax^3+bx^2+cx+d$ show that $P(x)-P(\gamma)$ has a factor $(x-\gamma)$. **2**

(ii) Hence show that the polynomial $P(x)-P(\gamma)$ has three distinct roots if **2**

$$(b-3a\gamma)(b+a\gamma)-4ac>0$$

Solutions HSC Ext 1 assessment Term 4 2013

1. $P(4) = 0 \Rightarrow 64a - 7\times16 + 4k + 4 = 0$ $[\div 4]$

$16a + k - 27 = 0$

$\therefore 16a + k = 27$

$P(1) = -6 \Rightarrow a - 7 + k + 4 = -6$

$a + 6 = -3$

(D)

2. $ts = -4$

$\therefore s = -\frac{4}{t}$

Midpoint $\left(\frac{2t+2s}{2}, \frac{t^2+s^2}{2}\right) = \left(t+s, \frac{t^2+s^2}{2}\right)$

$= \left(t - \frac{4}{t}, \frac{t^2 + \frac{16}{t^2}}{2}\right)$

$= \left(\frac{t^2-4}{t}, \frac{t^4+16}{2t^2}\right)$

(B)

3. $f'(x) = \frac{x^2}{x-2}$

x	-1	0	1
$f'(x)$	$-1/3$	0	-1

(A)

Horizontal pt of inflexion at $x=0$

4. $x^2 = 6y \Rightarrow 4a = 6$

$2a = 3$

Chord of contact $xx_0 = 3(y + y_0)$

$xx_0 - 3y - 3y_0 = 0$

(B)

$\therefore x_0 = 2$ and $y_0 = 1$ to match with $2x - 3y - 3 = 0$

5. $\frac{d}{dx}\left(\frac{x}{\sqrt{1-2x}}\right) = \frac{vu' - uv'}{v^2}$

$u = x$ $\quad v = (1-2x)^{\frac{1}{2}}$

$u' = 1$ $\quad v' = \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2)$

$= -\frac{1}{\sqrt{1-2x}}$

$= \frac{\sqrt{1-2x} - x \times \frac{-1}{\sqrt{1-2x}}}{1-2x}$

$= \frac{(1-2x)+x}{(1-2x)\sqrt{1-2x}} = \frac{1-x}{(1-2x)\sqrt{1-2x}}$

(A)

Question 6

6 a) Roots $1,\ 2+\sqrt{2},\ 2-\sqrt{2}$

$\therefore P(x) = (x-1)(x^2-(\alpha+\beta)x+\alpha\beta)$ where $\alpha+\beta = 2+\sqrt{2}+2-\sqrt{2}$
$= 4$

$= (x-1)(x^2-4x+2)$ $\quad \alpha\beta = (2+\sqrt{2})(2-\sqrt{2})$
$= 4-2$

$= x^3-5x^2+6x-2$ $\quad = 2$

$b = -5$
$c = 6$
$d = -2$

6 b) $x^3 - x^2 - 4x = A(x-1)^3 + B(x-1)^2 + C$

Sub $x=1$: $-4 = C$

Coeff of x^3: $1 = A$

Sub $x=0$ $\quad 0 = -A+B+C$

$B = 5$

$x^3-x^2-4x = (x-1)^3+5(x-1)^2-4$

However this question has an error, because there are other solutions, depending on the method used. ALL possible correctly developed solutions earned FULL marks. Marks were lost for algebraic mistakes.

$A(x-1)^3 + B(x-1)^2 + C = A(x^3-3x^2+3x-1) + B(x^2-2x+1) + C$

$\therefore \quad x^3 - x^2 - 4x \quad = Ax^3 - x^2(3A-B) - x(-3A+2B) - A+B+C$

Coeff of x^3: $1 = A$

Coeff of x^2 $\quad 1 = 3A - B$

$\therefore B = 2$

Coeff of x Here is where the conflict is: $-3A+2B = 4$ doesn't agree

Equating constants: $0 = {}^-A + B + C$

$-1 = C$

The original question for the test was $A(x-1)^3 + B(x-1)^2 + C(x-1) + D$ which has unique solutions $A=1,\ B=2,\ C=-3,\ D=-4$

In our attempt to make the question easier by eliminating the $(x-1)$ term, we introduced an invalid identity. We apologise for the error. If, in the future, you believe there's an error in a question, please alert the supervisor at the time.

6 c) $x = 2at$ $\quad$ $y = at^2$

$\frac{dx}{dt} = 2a$ $\quad$ $\frac{dy}{dt} = 2at$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{2at}{2a}$

$= t$

$\therefore$ At $P(2ap, ap^2)$, gradient is p

Tangent: $y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

(ii) At T, $y=0$ $\Rightarrow$ $0 = px - ap^2$

$x = \frac{ap^2}{p}$

$= ap$

(ii) (cont) $\therefore$ T is $(ap, 0)$

(iii) Normal is $x + py = 2ap + ap^3$

At T, $x = ap$

At R, $x = ap$: $ap + py = 2ap + ap^3$

$py = ap + ap^3$

$y = a + ap^2$

$\therefore$ R is $(ap, a + ap^2)$

(iv) Let $X = ap$ and $Y = a + ap^2$

Sub $\frac{X}{a} = p$ into $Y = a + a\left(\frac{X}{a}\right)^2$

$= a + \frac{X^2}{a}$ ✓

$a(Y - a) = X^2$

$\therefore$ This is a parabola with vertex $(0, a)$ and ✓
focal length $\frac{a}{4}$

Directrix $= a - \frac{a}{4}$

$= \frac{3a}{4}$

Question 7

a) $x^3 - 2x^2 - 7x + 2 = 0$

Possible roots : $\pm 1, \pm 2$

$P(1) \neq 0$, $P(-1) \neq 0$

$P(2) = 8 - 8 - 14 + 2$
$\neq 0$

$P(-2) = -8 - 8 + 14 + 2$
$= 0$

$\therefore$ One root is -2

Alternatively

$$
\begin{array}{r}
x^2 - 4x + 1 \\
(x+2)\overline{\smash{)}\, x^3 - 2x^2 - 7x + 2} \\
\underline{x^3 + 2x^2 } \\
-4x^2 - 7x \\
\underline{-4x^2 - 8x } \\
x + 2
\end{array}
$$

$P(x) = (x+2)(x^2 + kx + 1)$ by inspection

$P(1) = -6 \quad \Rightarrow \quad -6 = (1+2)(1+k+1)$
$= 3(2+k)$
$2 + k = -2$
$k = -4$

$\therefore$ Quadratic is $x^2 - 4x + 1 = 0$

$$x = \frac{4 \pm \sqrt{16 - 4}}{2}$$
$$= 2 \pm \sqrt{3}$$

Alternative Solution Having found that one root is -2 then let other roots be α, β

$\alpha + \beta - 2 = +2$
$\therefore \alpha + \beta = 4$

$-2\alpha\beta = -2$
$\alpha\beta = 1$

$\alpha + \frac{1}{\alpha} = 4$

$\alpha^2 + 1 = 4\alpha$
$\alpha^2 + 4\alpha + 1 = 0$
$\alpha = \frac{-4 \pm \sqrt{12}}{2}$
$= -2 \pm \sqrt{3}$

$\therefore$ Roots are $-2, 2+\sqrt{3}, 2-\sqrt{3}$

b) $P(x) = -2x^3 + kx^2 - mx + 5$

$P'(x) = -6x^2 + 2kx - m$

If $P(x)$ is to have stationary points then

$-6x^2 + 2kx - m = 0$ must have 1 or 2 solutions

To have solutions, $\Delta \geq 0$

ie $(2k)^2 - 4(-6)(-m) \geq 0$

$4k^2 - 24m \geq 0$

$k^2 - 6m \geq 0$

c) $y = \dfrac{(x-1)^2}{x+1}$

(i) $\dfrac{dy}{dx} = \dfrac{(x+1)\,2(x-1) - (x-1)^2 \times 1}{(x+1)^2}$

$= \dfrac{2(x^2-1) - (x^2-2x+1)}{(x+1)^2}$

$= \dfrac{x^2+2x-3}{(x+1)^2}$

$= \dfrac{(x+3)(x-1)}{(x+1)^2}$

Stat pts occur ~~at~~ when $\dfrac{dy}{dx} = 0$, ie when $x = -3, 1$

x	-4	-3	-2		0	1	2
$\frac{dy}{dx}$	$5/9$	0	$-\frac{3}{1}$		-3	0	$5/9$

There's a discontinuity at $x = -1$

There is a maximum turning point at $(-3, -8)$ ~~and~~
and a minimum turning point at $(1, 0)$

$$\frac{(x-1)^2}{x+1} = \frac{x^2-2x+1}{x+1}$$

$$\begin{array}{r}
x-3 \\
x+1 \enclose{longdiv}{x^2-2x+1} \\
\underline{x^2+x} \\
-3x+1 \\
\underline{-3x-3} \\
4
\end{array}$$

$$\therefore \frac{(x-1)^2}{x+1} = x-3+\frac{4}{x+1}$$

Alternatively

$$RHS = \frac{(x+1)(x-3)+4}{x+1}$$

$$= \frac{x^2-2x-3+4}{x+1}$$

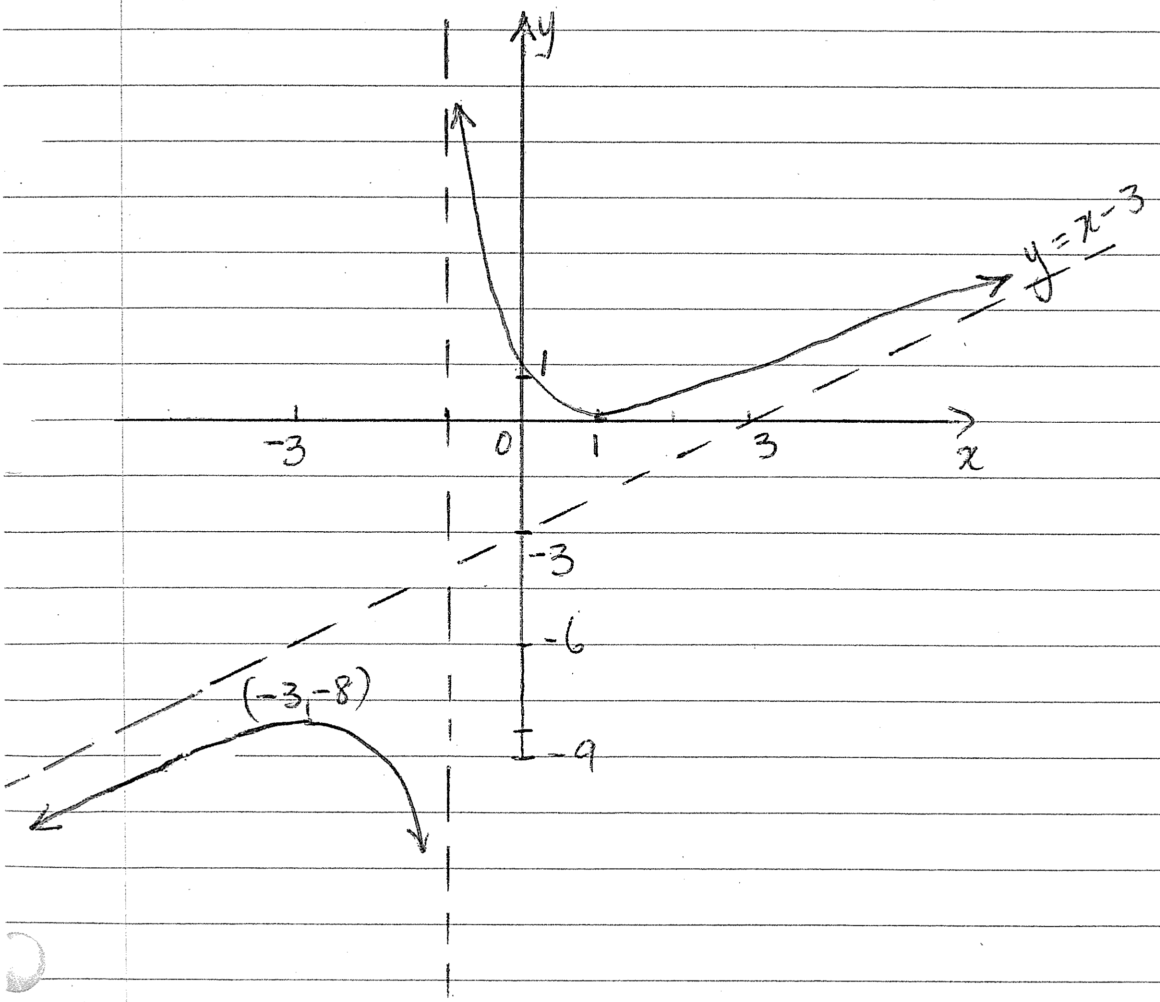
$$= \frac{x^2-2x+1}{x+1}$$

$$= \frac{(x-1)^2}{x+1}$$

$$= LHS$$

(iii) Intercept : $(1,0)$, $(0,1)$

Asymptotes : $x=-1$ and $y=x-3$



Question 8

(i) Point C (5,6)

(ii) $y = \frac{8x^2}{15} - \frac{31x}{15} + 3 \quad 0 \leq x \leq 5$

$\frac{dy}{dx} = \frac{16x}{15} - \frac{31}{15}$

Max gradient will occur when $x = 5$ due to ramp being parabolic

$\frac{dy}{dx} = \frac{16 \times 5}{15} - \frac{31}{15}$

$= \frac{49}{15}$

$= 3\,{}^{4}/_{15}$

(iii) Max gradient is 3 $\therefore \frac{16x}{15} - \frac{31}{15} = 3$

$x = 4 \cdot 75$

$\therefore y = \frac{8(4.75)^2}{15} - \frac{31 \times 4.75}{15} + 3$

$= 5 \cdot 21$ (2 d.p.

The level K can be set at 5.2 metres high.

b) $x = t^2 + 1$ and $y = t^{-1}$

$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = -\frac{1}{t^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= -\frac{1}{t^2} \times \frac{1}{2t}$

$= -\frac{1}{2t^3}$

At $t = -1$, $\frac{dy}{dx} = +\frac{1}{2}$

When $t=-1$, $x=2$ and $y=-1$

Equation of tangent: $y+1=\frac{1}{2}(x-2)$

$$2y+2=x-2$$

$\therefore x-2y-4=0$

c) $P(x)=ax^3+bx^2+cx+d$

$$P(x)-P(\gamma)=a(x^3-\gamma^3)+b(x^2-\gamma^2)+c(x-\gamma)+d-d$$

$$=a(x-\gamma)(x^2+x\gamma+\gamma^2)+b(x-\gamma)(x+\gamma)+c(x-\gamma)$$

$$=(x-\gamma)[ax^2+ax\gamma+a\gamma^2+bx+b\gamma+c]$$

$\therefore (x-\gamma)$ is a factor of $P(x)$

(ii) One root will be $x=\gamma$

Other roots will be roots of $(ax^2+x(a\gamma+b)+a\gamma^2+b\gamma+c)=0$

If two districts roots exist to this equation then $\Delta>0$

$$(a\gamma+b)^2-4a(a\gamma^2+b\gamma+c)>0$$

$$a^2\gamma^2+2ab\gamma+b^2-4a^2\gamma^2-4ab\gamma-4ac>0$$

$$b^2-3a^2\gamma^2-2ab\gamma-4ac>0$$

$$(b-3a\gamma)(b+a\gamma)=b^2+ab\gamma-3ab\gamma-3a^2\gamma^2$$

$$\therefore (b-3a\gamma)(b+a\gamma)-4ac>0.$$